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Dipole dynamics in the presence of a cosmic string

C A de Lima Ribeiro[1,3], **C Furtado**[1,2], **V B Bezerra**[2] **and F Moraes**[1]

[1] Laboratório de Física Teórica e Computacional, Departamento de Física, Universidade Federal de Pernambuco, 50670-901 Recife, PE, Brazil
[2] Departamento de Física, Universidade Federal da Paraíba, Caixa Postal 5008, 58051-970 João Pessoa, PB, Brazil

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Abstract

In this work we study the influence of a cosmic string on the classical and quantum dynamics of an electric dipole. We find the Lagrangian and the classical scattering. The Schrödinger equation for this problem is also solved and the quantum scattering is determined.

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1. Introduction

The quantum dynamics of a single particle in a conical space-time has been investigated by several authors. Deser and Jackiw [1] investigated the classical and quantum non-relativistic dynamics of a particle in a (2 + 1)-dimensional conical space-time. This problem was also investigated by 't Hooft [2]. In a recent article the influence of conical singularities in the energy levels of a harmonic oscillator was investigated [3].

It is well known that a charged particle in a conical space-time experiences a self-force of topological origin [4–7]. In this case the topology induces an effective interaction through the deformation of the Coulomb field. Gibbons *et al* [8] investigated the quantum dynamics of a charged particle in the presence of a cosmic string, considering the self-force in the dynamics. The condensed matter analogue of this problem was solved by Furtado and Moraes [9].

Using a curved space approach Azevedo and Moraes [10] recently obtained explicitly the scattering amplitude and total cross section for the quantum scattering by disclinations in a graphite sheet. This kind of model can provide an interesting way to investigate geometrical and topological effects.

In this paper we investigate the quantum and classical dynamics of a single electric dipole in a (2 + 1)-dimensional conical space-time. We introduce in the dynamics the self-force experienced by the dipole. The self-interaction is treated using the approach of Grats and Garcia [11], in which they considered the context of classical field theory, and obtained the

[3] Present address: Departamento de Física, Universidade Estadual de Feira de Santana, BR116-Norte, Km 03, Feira de Santana, Bahia, Brazil.

topological self-energy of a point electric dipole in (2 + 1)-dimensional conical space-time. Initially, we determine the Lagrangian of the problem, solve the Euler–Lagrange equations and find the trajectories. We shall treat the scattering of a dipole in the field of a disclination as a stationary point mass. We write the Schrödinger equation in this background, obtain the wavefunction and analyse the quantum scattering problem in Adhikari's approach [12].

2. Classical scattering

In recent works about the space-time produced by an infinite straight cosmic string, Linet and Smith [4, 5] have shown that a charged point particle becomes subject to a finite repulsive electric self-force. The particle feels a distortion in its field which is produced by the global properties of space. De Mello *et al* [7] have also determined the self-energy of a linear charge distribution in the same space-time and analysed the presence of magnetic and electric linear sources. Souraeep and Sahni [6] have calculated the Green function in a two-dimensional conical space-time. In condensed matter physics, Azevedo *et al* [13] have studied a model to describe charge localization around a disclination in mono-layer graphite. In this paper we discuss the topological self-energy for a point electric dipole in the presence of a cosmic string. For simplicity we study the problem in a z = constant section of space. This dipole is placed in a conical space which can be described from a geometrical point of view. The line element in the section z = constant is given by the metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}\rho^2 + \frac{\rho^2}{p^2}\,\mathrm{d}\theta^2 \tag{1}$$

in polar coordinates. In the metric above, we have that $\rho \geqslant 0$ is measured from the top of the cone. The angular range is $0 \leqslant \theta \leqslant 2\pi$ and gives its angular position around the singularity. The parameter p is related to the linear mass density μ of the cosmic string by $p^{-1} = 1\text{–}4\,\mathrm{G}\mu$. However, in the cosmic string context $p > 1$, in solids it can also assume values less than 1. This metric has a cone-like singularity at $\rho = 0$. In other words, the curvature tensor of the metric (1), considered as a distribution, is of the form

$$R_{12}^{12} = 2\pi(p^2 - 1)\delta^{(2)}(\rho). \tag{2}$$

Now we are going to determine the topological contribution to the self-energy. For this we follow Grats and Garcia [11]. They showed how to find the topological self-energy for a point dipole in the presence of a cone. The total energy of the system is obtained by summing the self-energies of the two charges and taking the coincidence limit, which results in

$$U_t = -\frac{2(\vec{n}\cdot\vec{d})(\vec{n}\cdot\vec{D}) - (\vec{d}\cdot\vec{D})}{\rho^2} \tag{3}$$

for the self-energy of the dipole. Here,

$$\vec{D} = -\frac{(p^2 - 1)}{48\pi}\vec{d} \tag{4}$$

where $\vec{d}$ is the dipole moment. The vector $\vec{n}$ is a unit Cartesian vector directed outward from the singularity. It means that the topological self-energy for a point dipole can be written as

$$U_t = \left(\frac{p^2 - 1}{48\pi}\right)\frac{d^2\cos(2\phi)}{\rho^2} \tag{5}$$

where ϕ is the angle between the dipole direction and the normal vector from the singularity.

From the above expression we observe that both states of minimal energy are stable. Any little deflection from equilibrium leads to a non-vanishing force moment which tends to decrease the deflection. The self-energy produces a minimal force when the dipole momentum is oriented in the perpendicular direction to the straight line connecting the top of the cone and the point where the dipole is placed.

In classical mechanics we have the following Lagrangian:

$$\mathcal{L} = \frac{\mu}{2}\dot{r}^i g_{ij}\dot{r}^j - U. \tag{6}$$

Substituting the topological self-energy and the kinetic energy into equation (6), we get

$$\mathcal{L} = \frac{\mu}{2}\left[\dot{\rho}^2 + \frac{\rho^2}{p^2}\dot{\theta}^2\right] + \frac{I}{2}\dot{\phi}^2 - \frac{(p^2-1)}{48\pi\rho^2}d^2\cos(2\phi) \tag{7}$$

where I is the dipole's momentum of inertia. The classical Lagrange equations of motion, in this case, are given by

$$\mu\ddot{\rho} = \mu\rho\frac{\dot{\theta}^2}{p^2} + \frac{(p^2-1)}{24\pi\rho^3}d^2\cos(2\phi) \tag{8}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\mu\rho^2\dot{\theta}}{p^2}\right] = 0 \tag{9}$$

$$\rho^2\ddot{\phi} = \frac{(p^2-1)}{24\pi I}d^2\sin(2\phi). \tag{10}$$

Note that we have a coupling between the variables. Therefore we cannot use the separation of variables method to solve these equations. Our development will take into account a particular position in which we have a minimum for the potential energy. The angular position corresponding to this situation is $\phi = \frac{\pi}{2}$. Choosing this, we have a simplification, and as a consequence, the equations of motion turn into

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mu\rho^2\dot{\theta}}{p^2}\right) = 0 \tag{11}$$

and

$$\mu\frac{\mathrm{d}\dot{\rho}}{\mathrm{d}t} = \frac{\mu\dot{\theta}^2}{p^2}\rho - \frac{(p^2-1)d^2}{24\pi\rho^3}. \tag{12}$$

Equation (11) gives

$$\dot{\theta} = \frac{mp^2}{\mu\rho^2} \tag{13}$$

where m is an integration constant. The radial equation is given by

$$\frac{\mathrm{d}^2\rho}{\mathrm{d}t^2} - \left(\frac{m^2}{\mu^2} - \frac{(p^2-1)d^2}{24\pi\mu}\right)\frac{1}{\rho^3} = 0. \tag{14}$$

Defining

$$B \equiv -\left[\frac{m^2}{\mu^2} - \frac{(p^2-1)d^2}{24\pi\mu}\right] \tag{15}$$

and performing some manipulations, we find

$$\dot{\rho}^2 + \frac{B}{\rho^2} = E \tag{16}$$

where E is a constant which can be related to the total energy of this system. From equation (16) we obtain the following result:

$$\rho^2 = \frac{B}{E} + (t - t_0)^2 \tag{17}$$

which can be rewritten as

$$\rho^2 = \frac{m^2}{E\mu^2} - \frac{(p^2-1)d^2}{24\pi E\mu} + E(t-t_0)^2. \tag{18}$$

For the angular dependence, we have

$$\theta - \theta_0 = \frac{(mp^2)}{(E\mu)}\frac{1}{\chi}\arctan\left[\frac{t-t_0}{\chi}\right] \tag{19}$$

where χ is given by

$$\chi^2 = \frac{m^2}{E^2\mu} - \frac{(p^2-1)d^2}{24\pi\mu E^2}. \tag{20}$$

Note that the classical scattering angle is obtained from the expression

$$\omega = \theta_{\text{out}} - \theta_{\text{in}} - \pi. \tag{21}$$

If we consider the particle moving from $t = -\infty$ to $+\infty$, the scattering angle will have the value

$$\omega = \pi\left[\frac{p^2}{[1-(\frac{p^2-1}{24\pi})d^2\,\mu\text{m}^2]^{\frac{1}{2}}} - 1\right]. \tag{22}$$

We note that if there is no dipole, we will re-obtain the result of Deser and Jackiw [1], and when the parameter p assumes the value 1 we have the flat space (no defect). Therefore we have no scattering due to the fact that in this case we have a locally and globally flat space-time.

3. The Schrödinger equation and quantum scattering

Now let us analyse the quantum behaviour of an electrical dipole in conical space. We are interested in solving the wave equation that describes the quantum dynamics of this system. We obtain the Schrödinger equation for this problem and look for its analytical solutions. Starting with the solution, we can analyse the quantum scattering. We will discuss the dipole behaviour in the presence of a conical singularity in two dimensions. We construct the kinetic operator from the symmetric metric tensor using the Laplace–Beltrami operator. The metric is given by (see equation (1))

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{p^2}{\rho^2} \end{bmatrix}. \tag{23}$$

The Hamiltonian is

$$H = -\frac{\hbar^2}{2\mu}\nabla^2_{\text{LB}} + V(\rho) \tag{24}$$

where ∇^2_{LB} is the Laplace–Beltrami operator and is given by the expression

$$\nabla^2_{\text{LB}} = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{p^2}{\rho^2}\frac{\partial^2}{\partial\theta^2} \tag{25}$$

and the interaction is the same as before:

$$V(\rho) = \frac{(p^2-1)}{48\pi}\frac{d^2}{\rho^2}. \tag{26}$$

The Schrödinger equation reduces to

$$\frac{1}{\rho}\partial_\rho(\rho\partial_\rho\Psi(\rho,\theta)) + \frac{p^2}{\rho^2}\frac{\partial^2}{\partial\theta^2} + \left[\frac{2\mu E}{\hbar^2} - \frac{(p^2-1)}{48\pi}\frac{d^2}{\rho^2}\right]\Psi(\rho,\theta) = 0. \tag{27}$$

We assume rotational symmetry and solve the above equation using the following ansatz:

$$\Psi(\rho, \theta) = F(\rho)\mathrm{e}^{\mathrm{i}l\theta} \tag{28}$$

where l is the orbital angular-momentum quantum number. Then, the equation for the radial coordinate is given by

$$\frac{\mathrm{d}^2 F(\rho)}{\mathrm{d}\rho^2} + \frac{1}{\rho}\frac{\mathrm{d}F(\rho)}{\mathrm{d}\rho} + \left\{\frac{2\mu E}{\hbar^2} - \frac{[l^2 p^2 + \frac{(p^2-1)}{48\pi}d^2]}{\rho^2}\right\} F(\rho) = 0. \tag{29}$$

Defining

$$\nu^2 = l^2 p^2 + \frac{(p^2-1)}{48\pi}d^2 \tag{30}$$

and

$$k^2 = \frac{2\mu E}{\hbar^2} \tag{31}$$

we identify equation (29) as a Bessel equation, whose solution is

$$F(\rho) = A_l J_\nu(k\rho) + B_l N_\nu(k\rho) \tag{32}$$

where we have two arbitrary constants and $J_\nu(k\rho)$ is the standard Bessel function and $N_\nu(k\rho)$ is the Neumann function. Imposing the regularity condition at the origin, we have to take B_l equal to zero.

In order to analyse the scattering problem we are going to use the standard decomposition of the wavefunction into the incident wave and the scattered wave. As concerning the problem of quantum scattering on a cone, there are two possibilities to choose an incoming state [8]. In what follows we will choose the incoming state as

$$\Psi_{\mathrm{in}}(\rho, \theta) = \mathrm{e}^{\mathrm{i}k\rho\cos\theta} \tag{33}$$

which is compatible with the system of coordinates which we are using in this paper. Therefore, we can write the wavefunction in terms of the incident and scattered waves as

$$\Psi(\rho, \theta) \to \mathrm{e}^{\mathrm{i}k\rho\cos\theta} + \Psi_{\mathrm{sc}}(\rho, \theta). \tag{34}$$

Using the expansion

$$\mathrm{e}^{\mathrm{i}k\rho\cos\theta} = \sum_{l=-\infty}^{\infty} i^l J_l(k\rho)\mathrm{e}^{\mathrm{i}l\theta} \tag{35}$$

and the asymptotic expansion for J_l, we find that

$$\Psi(\rho, \theta) \to \sum_{l=-\infty}^{\infty} i^l \left(\frac{2}{\pi k\rho}\right)^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}l\theta} \cos\left[k\rho - \left(|\nu| + \frac{1}{2}\right)\frac{\pi}{2}\right] + \left(\frac{\mathrm{i}}{k\rho}\right)^{\frac{1}{2}} f(\theta)\mathrm{e}^{\mathrm{i}\rho}. \tag{36}$$

Therefore, the scattering amplitude is given by

$$f(\theta) = \sum_{l=-\infty}^{\infty} \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}l\theta}\mathrm{e}^{\mathrm{i}\frac{\pi}{2}(l-\nu)} \sin\left[\frac{\pi}{2}(l-\nu)\right]. \tag{37}$$

If we intend to compare this with the optical theorem we need

$$\mathrm{Im}\,[f(\theta = 0)] = \mathrm{Im}\left\{\sum_{l=-\infty}^{\infty} \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}\frac{\pi}{2}(l-\nu)} \sin\left[\frac{\pi}{2}(l-\nu)\right]\right\} \tag{38}$$

then we have

$$\mathrm{Im}\,[f(0)] = \sum_{l=-\infty}^{\infty} \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sin^2\left[\frac{\pi}{2}(l-\nu)\right]. \tag{39}$$

Evaluating the cross section from

$$\sigma = \int_0^{2\pi} \frac{\sigma(\theta)}{p}\,\mathrm{d}\theta \tag{40}$$

we get

$$\sigma = \frac{4}{kp}\sum_{l=-\infty}^{\infty} \sin^2\left[\frac{\pi}{2}(l-\nu)\right]. \tag{41}$$

Expanding the square root in equation (30) and obtaining its first and second terms, we have

$$\nu \cong lp\left[1 + \frac{(p^2-1)d^2}{96\pi l^2 p^2}\right]. \tag{42}$$

Using this result our equation turns into

$$f(\theta) = \sum_{l=-\infty}^{\infty} \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}l\theta+\delta_l} \sin\delta_l \tag{43}$$

where the phase shift δ_l is given by

$$\delta_l = \frac{\pi}{2}\left[l(1-p) - \frac{(p^2-1)d^2}{96\pi lp}\right]. \tag{44}$$

Therefore the cross section is

$$\sigma = \frac{4}{kp}\sum_{l=-\infty}^{\infty} \sin^2\left\{\frac{\pi}{2}\left[l(1-p) - \frac{(p^2-1)d^2}{96\pi lp}\right]\right\} \tag{45}$$

and

$$\mathrm{Im}\,(f(0)) = \sum_{l=-\infty}^{\infty} \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sin^2\left\{\frac{\pi}{2}\left[l(1-p) - \frac{(p^2-1)d^2}{96\pi lp}\right]\right\}. \tag{46}$$

Note from above expressions (45), (46) that we verify the optical theorem in Adhikari's approach [12].

4. Concluding remarks

In this work, we have analysed how to treat an electric dipole in the conical space-time of a cosmic string, which has a analogue in condensed matter physics, a medium with a disclination. We found the equations of motion from the Lagrangian and observed that our equations are coupled. We found analytical solutions in the classical and quantum approaches. We have shown that if the parameter p is different from unity, we will have classical scattering. We have also found the scattering amplitude in the quantum case and verified the optical theorem, where the imaginary part of the amplitude was related to the total cross section.

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